SPINOR EQUATIONS IN WEYL GEOMETRY[1]

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Abstract: In this paper, the Dirac, twistor and Killing equations on Weyl manifolds with CSpin structures are investigated. A conformal Schrödinger-Lichnerowicz formula is presented and used to show integrability conditions for these equations. By introducing the Killing equation for spinors of arbitrary weight, the result of Andrei Moroianu in [9] is generalized in the following sense. The only non-closed Weyl manifolds of dimension greater than 3 that admit solutions of the real Killing equation are 4-dimensional and non-compact. Any Weyl manifold of these dimensions admitting a real Killing spinor has to be Einstein-Weyl.

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0 Introduction

In the first section, we state some basic definitions of density bundles, Weyl structures, curvature terms and Einstein-Weyl structures.

The second section is dedicated to Dirac- and twistor operators on Weyl manifolds. In [3], [1], [7] and [8] the properties and integrability conditions of the twistor and Killing equation were intensively studied in the context of Riemannian geometry. Here we want to generalize some of these results to arbitrary Weyl structures and spinor fields of arbitrary weight. The first result in this area is due to Andrei Moroianu [9] and deals with the integrability conditions for the existence of non-trivial parallel spinors of weight 0. He found that the given Weyl structure has to be flat (closed) on manifolds, which are not 4-dimensional and non-compact. Furthermore, he gave several counter examples by showing, that in dimension 4 the existence of a parallel spinor field is equivalent to the existence of a hypercomplex structure. This means in particular (see [10]), that the Weyl structure is Einstein-Weyl. We generalize this result to any dimension $n > 2$ as well as for Killing spinor fields ψ of arbitrary weight satisfying

$$\nabla^S_X \psi = \beta X \cdot \psi,$$

where β denotes a complex density of weight -1. The only non-closed Weyl manifolds of dimension greater than 3 that admit solutions of the real Killing equation are 4-dimensional and non-compact.

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Any Weyl manifold of these dimensions admitting a real Killing spinor has to be Einstein-Weyl. To this end it is crucial to proof a generalized Schrödinger-Lichnerowicz formula:

$$\mathcal{D}^2\psi = \Delta^S\psi + \frac{1}{4}R\psi + \left(\frac{n-2+2w}{4}\right) F\cdot\psi,$$

where R denotes the scalar curvature and F the Faraday curvature of the Weyl structure.
This formula is also used in order to investigate integrability conditions of the twistor equation, which is defined by

$$0 = \mathcal{T}_W\psi := \nabla^S\psi + \frac{1}{n}\nu\mathcal{D}\psi.$$

We then compute

$$\nabla^S\mathcal{D}\psi = \frac{n}{n-2}\left[-\frac{1}{2}\mu^2 Ric' + \frac{1}{4(n-1)}R\nu + \left(w-\frac{1}{2}\right)\left(\mu^2 F + \frac{2}{4(n-1)}\nu\mu F\right)\right]\psi$$

on its kernel, where Ric' is the Ricci curvature of the $\mathfrak{o}(n)$-component W' of the Weyl structure W. This equation corresponds to the equation $\nabla^S_X\mathcal{D}\psi = \frac{n}{2(n-2)}\left(\frac{R}{2(n-1)}X - Ric(X)\right)\cdot\psi$ in [3]. We use this result to prove, that the two well known first integrals $C(\psi)$ and $Q(\psi)$ are parallel densities if the weight of ψ is $\frac{1}{2}$ or $d\theta\cdot\psi = 0$. Furthermore, we use it in order show that the zeros of a twistor spinor field form a discrete set.

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1 Weyl geometry on conformal Spin manifolds

Let M^n be a smooth, oriented manifold and $(\mathbf{R}, M^n, \pi, GL(n,\mathbb{R})$ its frame bundle. Let $CO(n)_+ = SO(n)\times\mathbb{R}_+$. For a conformal class c let $\mathbf{P}$ denote the corresponding $CO(n)_+$-reduction. We define a two-fold covering $\lambda^c : Spin(n)\times\mathbb{R}_+ =: CSpin(n) \longrightarrow CO(n)_+ = \{A\in CO(n)|det(A)>0\}$ by

$$\lambda^c(a,\vartheta) := \vartheta\lambda(a),$$

$\lambda : Spin(n) \longrightarrow SO(n)$ is the covering of the $SO(n)$. The spinor representation κ^w of $CSpin(n)$ on $\Delta_n := \mathbb{C}^{2^{\left[\frac{n}{2}\right]}}$ with weight w is defined as follows: $\kappa^w(a,\vartheta) = \vartheta^w\kappa(a)$, where κ is the $Spin(n)$-representation on Δ_n. Like a Spin structure a CSpin structure on (M^n, c) is a pair $(\mathbf{P}_{CSpin}, \Lambda^c)$, where
$(\mathbf{P}_{CSpin}, \pi_{CSpin}, M^n, CSpin(n))$ is a $CSpin(n)$-principal fibre bundle on M^n and
$\Lambda^c : \mathbf{P}_{CSpin} \longrightarrow \mathbf{P}_{CO_+}$ is a two-fold covering that commutes with λ^c and the action of the structure group. The existence of $CSpin$ structures is equivalent to the existence of $Spin$ structures, since $Spin(n)$ is maximally compact in $CSpin(n)$. We have the following vector bundles:

1. $\mathcal{L}^w := \mathbf{P}_{CSpin}\times_{|det\circ\lambda^c|^{\frac{w}{n}}}\mathbb{R}$ is called density bundle with weight w.

2. $(T^{r,s})^w := \mathbf{P}_{CSpin}\times_{(\rho^{r,s}\circ\lambda^c)^w}(\bigotimes^r(\mathbb{R}^n)^*\bigotimes^s\mathbb{R}^n)$, is the (r,s)-Tensor bundle with weight w. $(\rho^{r,s}\circ\lambda^c)^w$ denotes the the standard representation of $CSpin(n)$ on the (r,s)-tensors with weight w, $(\rho^{r,s}\circ\lambda^c)^w(a,\vartheta) = \vartheta^w(\rho^{r,s}\circ\lambda^c)(a)$. T shall denote the ordinary tagent bundle and T^* its dual.

3. $S^w := \mathbf{P}_{CSpin}\times_{\kappa^w}\Delta_n$ is the spinor bundle with weight w. $S := S^1$ denotes the ordinary spin bundle,

Let $|vol_g|^{-\frac{1}{n}} =: l_g \in \mathcal{L}^1$ denote the density corresponding to a metric $g\in c$. Then we are given the following conformally invariant operators

1. $c := l_g^2 g : T^w \otimes T^{w_1} \longrightarrow \mathcal{L}^{w+w_1}$, $|X|^2 := c(X,X) \in \mathcal{L}^{2w}$ for $X \in T^w$.
2. $(.)_c := l_g^2 (.)_g : T^w \longrightarrow (T^*)^w$,
3. $tr := l_g^{-2} tr_g : (T^{r,s})^w \longrightarrow (T^{r-2,s})^w$, $r \geq 2$.
4. the conformal, hermitian product $(.,.) := l_g^2 (.,.)_g : \Gamma(S^w \otimes S^{w_1}) \longrightarrow \Gamma(\mathcal{L}^{w+w_1})$ and
5. the conformal Clifford product $\mu := l_g \mu_g : \Gamma(T^w \otimes S^{w_1}) \longrightarrow \Gamma(S^{w+w_1})$,

where $(.,.)_g$ and μ_g denote the hermitian product and the Clifford product given on $(M^n, g \in c)$. We can use $(.)_c$ in order to define the μ on arbitrary (r,s)-tensor fields. The operator $\mu^{ab} : T^{r,s} \otimes S^w \longrightarrow S^w$ is the conformal Clifford product of a spinor field of weight w with the b^{th} and then with the a^{th} component of a tensor field. Example:

$$\mu^{21} \gamma \otimes X \otimes \omega \otimes \psi = \omega \otimes X \cdot \gamma \cdot \psi,$$

where $\gamma \otimes X \in (T^{1,1})^w$, $\omega \in T^{2,0}$ and $\psi \in \Delta_n^{w_1}$. Whenever there are no indices, the Clifford product ranges over all components of the corresponding tensor, i.e. $\mu A \otimes \psi = A \cdot \psi := \sum_{i_1,\dots,i_r} A(e_{i_1},\dots,e_{i_r}) e_{i_1} \cdots e_{i_r} \cdot \psi$. The operator $\nu : \Delta_n^w \longrightarrow T^{1,0} \otimes \Delta_n^w$ is defined as follows:

$$X \rfloor \nu \psi = \mu X \otimes \psi = X \cdot \psi.$$

Then $\mu\nu = -n$ holds. Some well known identities have then the following appearance:

$$\mu^{12} \omega \otimes \psi = -\mu^{21} \omega \otimes \psi - 2 tr^{12} \omega \psi \tag{1}$$
$$tr \nu \omega \psi = w \cdot \psi \tag{2}$$
$$Re(\nu\psi, \nu\psi) = (\psi, \psi) c := |\psi|^2 c \tag{3}$$

Moreover, we define some operators on $T^{r,s}$:

1. Let (ab) denote the transposition of the components a and b,
2. $Sym := Id + (12)$, $Alt = Id - (12)$, $Zyk := Id + (23)(12) + (12)(23)$, $Zyk^{1234} := Id + (12)(23)(34) + (34)(23)(12) + (13)(24)$.

A torsion-free connection $W : T\mathbf{P} \longrightarrow \mathfrak{co}(n)$ on a conformal manifold (M^n, c) is called Weyl structure. ∇ shall denote the induced covariant derivatives on associated vector bundles. The operators c, tr and $(.)_c$ are parallel with respect to any Weyl structure. On $\mathcal{L}^1$ the curvature of a Weyl structure is given by $Alt \nabla^{T^* \otimes \mathcal{L}^1} \nabla^{\mathcal{L}^1} =: F \in \Omega^2(M)$. This globally defined 2-form is called Faraday curvature. Choosing a gauge g on M^n provides a 1-form $\theta \in \Omega^1(M^n)$ in the following way: $\nabla l_g = \theta \otimes l_g$. For any gauge, we obtain $F = d\theta$. Since the Lie algebra of the conformal group splits into two components, there is also a splitting of a Weyl structure into a metric part W' and a scalar part $\theta' \otimes Id$.

$$W = W' + \theta' \otimes Id, \quad W' : T\mathbf{P} \longrightarrow \mathfrak{o}(n), \quad \theta' \in \Omega^1(P)$$

A Weyl structure W is exact (closed) if and only if θ is exact (closed) with respect to any gauge. For a given Weyl manifold (M^n, c, W) the curvature tensor $\mathcal{R} \in \Gamma(T^{4,0})^{-2}$ is defined by

$$\mathcal{R}(X,Y,Z,U) := c(\nabla_X \nabla_Y U - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, U),$$

for any vector fields $X, Y, Z, U \in \Gamma(T)$. The Ricci curvature is given by:

$$Ric := tr^{14} \mathcal{R} \text{ and } Ric' := tr^{14} \mathcal{R}' = Ric + F,$$

where the primed objects belong to the connection W'. Ric is *not* symmetric. In fact, we obtain: $\frac{1}{2} Alt Ric = -\frac{n}{2} F$. Finally, we define the scalar curvature

$$R := tr(Ric) = tr tr^{14} \mathcal{R} \in \Gamma(\mathcal{L}^{-2}),$$

which is not a function, but a density of weight -2.

Lemma 1.1 (Symmetry properties) *Let $\mathcal{R}$' be the curvature tensor of W'. Then*

$$\mathcal{R}' = (13)(24)\mathcal{R}' + [(13) + (23) - (14) - (24)]\, F \otimes c. \tag{4}$$

Proof: We have $Zyk\mathcal{R}' = -ZykF \otimes c$, which is just a version of the first Bianchi identity for $\mathcal{R}$. This yields:

$$Zyk\mathcal{R}' Zyk^{1234} = -ZykF \otimes cZyk^{1234}.$$

We choose vector fields X, Y, Z, T. Then:

$$\begin{aligned} Zyk\mathcal{R}' Zyk^{1234}(X,Y,Z,T) &= \mathcal{R}'(X,Y,Z,T) + \mathcal{R}'(Y,Z,X,T) + \mathcal{R}'(Z,X,Y,T) \\ + &\quad \mathcal{R}'(Y,Z,T,X) + \mathcal{R}'(Z,T,Y,X) + \mathcal{R}'(T,Y,Z,X) + \mathcal{R}'(Z,T,X,Y) + \mathcal{R}'(T,X,Z,Y) \\ + &\quad \mathcal{R}'(X,Z,T,Y) + \mathcal{R}'(T,X,Y,Z) + \mathcal{R}'(X,Y,T,Z) + \mathcal{R}'(Y,T,X,Z) \\ = &\quad 2\left(\mathcal{R}'(Z,X,Y,T) + \mathcal{R}'(T,Y,Z,X)\right) = 2(12)(23)(\mathcal{R}' - (13)(24)\mathcal{R}')(X,Y,Z,T). \end{aligned}$$

Similarly, we get for $F \otimes c$: $ZykF \otimes cZyk^{1234} = 2Zyk^{1234}F \otimes c$. Putting all this together implies:

$$\mathcal{R}' = (13)(24)\mathcal{R}' - (12)(23)Zyk^{1234}F \otimes c = (13)(24)\mathcal{R}' + [(13) + (23) - (14) - (24)]F \otimes c,$$

since

$$\begin{aligned} (12)(23)Zyk^{1234}F \otimes c &= (12)(23)[Id + (12)(23)(34) + (34)(23)(12) + (13)(24)]F \otimes c \\ &= [-(13) - (13)(23)(12) + (14) + (23)(24)(34)]F \otimes c \\ &= [-(13) - (23) + (14) + (24)]F \otimes c. \end{aligned}$$

□

For $n \geq 3$ a Weyl structure W on (M^n, c, W) is said to be Einstein-Weyl if and only if

$$Ric = \frac{R}{n} \cdot c - \frac{n}{2}F \quad or \quad Ric' = \frac{R}{n} \cdot c - \frac{n-2}{2}F. \tag{5}$$

The symmetric part of Ric reduces to its trace if and only if W is Einstein-Weyl. (M^n, c, W) is called an Einstein-Weyl manifold.
Let W^S be the lift of W into the $CSpin$ structure and denote its induced covariant derivative on S^w by ∇^S.

Theorem 1.2 [6] *Fix a gauge $g \in c$ and a spinor $\psi \in \Gamma(S^w)$. Then the difference between the spinor derivatives of W and W^g, the Levi-Civita connection is as follows:*

$$\nabla^S_X \psi - \nabla^{S,g}_X \psi = -\frac{1}{2} X \cdot \theta \cdot \psi + \left(w - \frac{1}{2}\right) \theta(X)\psi.$$

We define the spinorial curvature by $\mathcal{R}^{S,w} := Alt\nabla^{T^* \otimes S} \circ \nabla^S = \kappa^w_* \Omega^S$, where Ω^S is the curvature form of W^S.

Lemma 1.3

$$\begin{aligned} \mathcal{R}^{S,w} &= \frac{1}{4}\mu^{34}\mathcal{R}' + wF & (6) \\ \mu^{234}\mathcal{R}' &= -2\mu^2 Ric' - 2\mu^2 F - \nu\mu F & (7) \\ \mu\mathcal{R}' &= 2R + 2(n-2)\mu F & (8) \end{aligned}$$

PROOF: (1) According to the splitting $W = W' + \theta' \otimes Id$ we get for the corresponding curvature form: $\Omega = \Omega' + F$.

$$\mathcal{R}^S = \kappa_*^w \Omega^S = \kappa_*^w (\lambda_*^c)^{-1}\Omega = \kappa_*^w (\lambda_*^c)^{-1}\Omega' + wF = \frac{1}{4}\mu^{34}\mathcal{R}' + wF'.$$

(2) First we use the symmetry properties of $F \otimes c$ and (1) to calculate:

$$\begin{aligned} &-\mu^{124}[(13)+(23)-(14)-(24)]F\otimes c \\ &= \left[-\mu^{324}-\mu^{134}+\mu^{124}(14)-\mu^{124}-2\mu^1 tr^{24}\right] F\otimes c \\ &= \left[2\mu^{234}+2\mu^2 tr^{23}-\mu^{214}(14)-2tr^{12}\mu^4(14)-\mu^{124}+2\mu^2 tr^{23}\right] F\otimes c \\ &= \left[2\mu^{234}+4\mu^2 tr^{23}+\mu^{214}+2\mu^1 tr^{14}-2\mu^1 tr^{23}-\mu^{124}\right] F\otimes c \\ &= \left[2\mu^{234}+4\mu^2 tr^{23}-\mu^{124}+4\mu^2 tr^{23}-\mu^{124}\right] F\otimes c \\ &= \left[2\mu^{234}+8\mu^2 tr^{23}-2\mu^{124}\right] F\otimes c = (-2n+8)\mu^2 F - 2F\cdot\nu \end{aligned}$$

and

$$\begin{aligned} &-\mu^{234} ZykF\otimes c = -\mu^{234}\left[Id+(23)(12)+(12)(23)\right] F\otimes c \\ &= -\left[\mu^{234}-\mu^{234}(23)+\mu^{124}\right] F\otimes c = -\left[2\mu^{234}+2\mu^2 tr^{23}+\mu^{124}\right] F\otimes c \\ &= (2n-2)\mu^2 F - F\cdot\nu. \end{aligned}$$

This implies, by using $F\cdot\nu = \nu\mu F + 4\mu^2 F$ and equation (4):

$$\begin{aligned} &\mu^{234}\mathcal{R}' = -\mu^{234}\left[\left[(12)(23)+(23)(12)\right]\mathcal{R}' - ZykF\otimes c\right] \\ &= \mu^{234}(23)\mathcal{R}' - \mu^{124}\mathcal{R}' - \mu^{234} ZykF\otimes c \\ &= \mu^{234}(23)\mathcal{R}' - \mu^{124}(13)(24)\mathcal{R}' - \mu^{124}[(13)+(23)-(14)-(24)]F\otimes c - \mu^{234} ZykF\otimes c \\ &= \mu^{234}(23)\mathcal{R}' + \mu^{324}\mathcal{R}' + 2\mu^2 tr^{24}\mathcal{R}' + (-2n+8)\mu^2 F - 2F\cdot\nu + (2n-2)\mu^2 F - F\cdot\nu \\ &= -2\mu^{234}\mathcal{R}' - 6\mu^2 Ric' + 6\mu^2 F - 3F\cdot\nu = -2\mu^{234}\mathcal{R}' - 6\mu^2 Ric' - 6\mu^2 F - 3\nu\mu F. \end{aligned}$$

(3) From (7), we obtain

$$\begin{aligned} \mu\mathcal{R}' &= \mu\mu^{234}\mathcal{R}' - 2\mu Ric' - \mu F + n\mu F = -\mu(Alt(Ric') + Sym(Ric')) + (n-2)\mu F \\ &= (n-2)\mu F + 2R + (n-2)\mu F = 2(n-2)\mu F + 2R. \end{aligned}$$

□

2 Spinor equations in Weyl geometry

2.1 The Dirac operator

Definition 2.1 (Dirac operator) *The Dirac operator* $\mathcal{D}_W : \Gamma(S^w) \longrightarrow \Gamma(S^{w-1})$ *is defined by:*

$$\mathcal{D}_W := \mu\nabla^S.$$

If there is no ambiguity to be expected, we will omit the index W.

Definition 2.2 *The spinor Laplacian is given by* $\Delta^{S,w} : \Gamma(S^w) \longrightarrow \Gamma(S^{w-2})$

$$\Delta^{S,w} := -tr\nabla^{T^*\otimes S} \circ \nabla^S.$$

Theorem 2.3 (Schrödinger-Lichnerowicz formula) *Let $\psi \in \Gamma(S^w)$. Then*

$$\mathcal{D}^2\psi = \Delta^S\psi + \frac{1}{4}R\psi + \left(\frac{n-2+2w}{4}\right) F \cdot \psi. \tag{9}$$

Proof: This can be obtained directly. The final reduction of the curvature terms is due to the equations (6) and (8).

$$\begin{aligned}
\mathcal{D}^2\psi &= \mu\nabla^S\mu\nabla^S\psi = \mu\nabla^{T^*\otimes S}\nabla^S\psi = \mu\frac{1}{2}(Alt\nabla^{T^*\otimes S}\nabla^S + Sym\nabla^{T^*\otimes S}\nabla^S)\psi \\
&= \mu\frac{1}{2}\mathcal{R}^S\psi - tr\nabla^{T^*\otimes S}\nabla^S\psi = \Delta^S\psi + \frac{1}{8}\mu\mathcal{R}'\psi + \frac{1}{2}wF\cdot\psi \\
&= \Delta^S\psi + \frac{1}{4}R\psi + \left(\frac{n-2}{4} + \frac{w}{2}\right) F \cdot \psi.
\end{aligned}$$

□

2.2 The twistor operator

Definition 2.4 (twistor operator) *We define the twistor operator $\mathcal{T}_W : \Gamma(S^w) \longrightarrow \Gamma(T^* \otimes S^w)$ of a CSpin manifold (M^n, c, W) by $\mathcal{T}_W := \nabla^{S,w} + \frac{1}{n}\nu\mathcal{D}$.*

Let (M^n, c, W) be a $CSpin$ manifold and $\psi \in \Gamma(S^w)$ a twistor spinor field, i.e. an element of the kernel of $\mathcal{T}$. Then $\nabla^S\psi = -\frac{1}{n}\nu\mathcal{D}\psi$ is true and therefore

$$\Delta^S\psi = -tr\nabla^{T^*\otimes S}\nabla^S\psi = \frac{1}{n}tr\nabla^{T^*\otimes S}\nu\mathcal{D}\psi = \frac{1}{n}tr\nu\nabla^S\mathcal{D}\psi = \frac{1}{n}\mathcal{D}^2\psi \tag{10}$$

is satisfied. From the Schrödinger-Lichnerowicz formula we obtain:

$$\mathcal{D}^2 = \frac{n}{4(n-1)}R + \frac{(n-2+2w)n}{4(n-1)}\mu F. \tag{11}$$

This leads to

Theorem 2.5 *Let $\psi \in \Gamma(S^w)$ be a twistor spinor. Then:*

$$\nabla^S\mathcal{D}\psi = \frac{n}{n-2}\left[-\frac{1}{2}\mu^2 Ric' + \frac{1}{4(n-1)}R\nu + \left(w - \frac{1}{2}\right)\left(\mu^2 F + \frac{1}{2(n-1)}\nu\mu F\right)\right]\psi. \tag{12}$$

Proof: We use (7) and (6) in the first and second step. Then finally, after some direct calculations, we use (11).

$$\begin{aligned}
-\frac{1}{2}&\mu^2 Ric'\psi - \frac{1}{2}\mu^2 F\psi - \frac{1}{4}\nu\mu F\psi = \frac{1}{4}\mu^{234}\mathcal{R}' \\
&= \mu^2\mathcal{R}^S\psi - w\mu^2 F\psi = \mu^2 Alt\nabla^{T^*\otimes S}\nabla^S\psi - w\mu^2 F\psi \\
&= -\frac{1}{n}\mu^2 Alt\nabla^{T^*\otimes S}\nu\mathcal{D}\psi - w\mu^2 F\psi = \frac{1}{n}\mu^2 Alt\nu\nabla^S\mathcal{D}\psi - w\mu^2 F\psi \\
&= \frac{1}{n}\mu^2\nu\nabla^S\mathcal{D}\psi - \frac{1}{n}\mu^1\nu\nabla^S\mathcal{D}\psi - w\mu^2 F\psi = -\frac{1}{n}\nabla^S\mathcal{D}^2\psi - \frac{2}{n}\nabla^S\mathcal{D}\psi + \nabla^S\mathcal{D}\psi - w\mu^2 F\psi \\
&= -\frac{n}{4n(n-1)}R\nu\psi - \frac{(n-2-2w)}{4(n-1)}\nu\mu F\psi + \frac{n-2}{n}\nabla^S\mathcal{D}\psi - w\mu^2 F\psi.
\end{aligned}$$

□

Theorem 2.6 *If the term* $\left(w-\frac{1}{2}\right)\left(\mu^2F+\frac{2}{4(n-1)}\nu\mu F\right)\psi$ *reduces to a* single *Clifford product or even vanishes, e.g. if* $w=\frac{1}{2}$ *or* $F\cdot\phi=0$ *is satisfied, the sections*

$$C(\psi) := Re(\psi,\mathcal{D}\psi)\in\Gamma(\mathcal{L}^{2w-1})$$

and

$$Q(\psi) := |\psi|^2|\mathcal{D}\psi|^2 - trRe(\mathcal{D}\psi,\nu\psi)^2\in\Gamma(\mathcal{L}^{4w-2})$$

are W-parallel.

Proof: In (12) there are only single Clifford products left. Then (3) yields

$$\nabla C(\psi) = Re(\nabla^S\psi,\mathcal{D}\psi)+Re(\psi,\nabla^S\mathcal{D}\psi) = Re(-\frac{1}{n}\nu\mathcal{D}\psi,\mathcal{D}\psi)=0$$

and

$$\begin{aligned}\nabla Q(\psi) &= 2Re(\nabla^S\psi,\psi)Re(\mathcal{D}\psi,\mathcal{D}\psi)+2Re(\psi,\psi)Re(\nabla^S\mathcal{D}\psi,\mathcal{D}\psi)\\ &\quad -2tr^{23}Re(\nabla^S\mathcal{D}\psi,\nu\psi)Re(\mathcal{D}\psi,\nu\psi)+\frac{2}{n}tr^{13}Re(\mathcal{D}\psi,\nu\nu\mathcal{D}\psi)Re(\mathcal{D}\psi,\nu\psi)\\ &= 0.\end{aligned}$$

□

2.2.1 The zeros of a twistor spinor field

In this section we show that the zeros of a twistor spinor field are a discrete set in M^n. Let M^n be connected. We define $E^w := S^w\oplus S^{w-1}$ and regard the covariant derivative ∇^{E^w}, which is characterized by

$$\nabla^{E^w} = \begin{pmatrix}\nabla^{S,w} & \frac{1}{n}\nu\\ K^w & \nabla^{S,w-1}\end{pmatrix},$$

where $K^w=\nabla^{S,w-1}\mathcal{D}:\Gamma(S^w)\longrightarrow\Gamma(S^{w-1})$.

Theorem 2.7 *For all twistor spinor fields* $\phi\in\Gamma(S^w)$

$$\nabla^{E^w}\begin{pmatrix}\phi\\ \mathcal{D}\phi\end{pmatrix}=0$$

holds. Conversely, any ∇^{E^w}*-parallel section* $\begin{pmatrix}\phi\\ \psi\end{pmatrix}\in\Gamma(E^w)$ *yields:*

$$\mathcal{T}_W\phi=0\quad\text{and}\quad\psi=\mathcal{D}\phi.$$

Since parallel sections on vector bundles over connected manifolds are uniquely determined by their value in a single point, we obtain:

Corollary 2.8 *The dimension of the space of all twistor spinor fields of connected CSpin manifold is less than or equal to* $2^{[\frac{n}{2}]+1}$*. Furthermore, a twistor spinor field* ψ *on a connected CSpin manifold for that* $\psi(m)=0$ *and* $\mathcal{D}\psi(m)=0$ *in a point* $m\in M^n$ *is trivial.*

Theorem 2.9 *The set* $N_\psi := \{\psi\in\Gamma(S^w):\ \psi(m)=0$ *and* $\mathcal{T}_W\phi=0\}$ *of a twistor spinor field* $0\neq\psi\in\Gamma(S^w)$ *is discrete in* M^n*.*

PROOF: (12) yields $\nabla^S \mathcal{D}\phi(m) = 0$ and for $g \in c$:

$$0 = 2Re(\nabla\psi, \psi)(m) = \nabla|\psi|^2(m) = \nabla^g|\psi|^2(m),$$

since $\nabla l^w = \nabla^g l^w + w\theta \otimes l^w$. $|\psi|^2$ is a density, i.e a section of $\mathcal{L}^{2w}$. Therefore, we get for vector fields X, Y on M^n:

$$\nabla_X \nabla_Y |\psi|^2(m) = \nabla^g_X \nabla_Y |\psi|^2(m) + 2w\theta(X)\nabla_Y|\psi|^2(m) = \nabla^g_X \nabla^g_Y |\psi|^2(m).$$

If we choose X and Y to be W-parallel in m, we finally obtain by applying (3)

$$\begin{aligned} \nabla_X \nabla_Y |\psi|^2(m) = 2\nabla_X(\nabla_Y\psi, \psi)(m) &= -\frac{2}{n}\nabla_X(Y \cdot \mathcal{D}\psi, \psi)(m) \\ = \quad \frac{2}{n^2}(Y \cdot \mathcal{D}\psi, X \cdot \mathcal{D}\psi)(m) &= \frac{2}{n^2}c(X, Y)|\mathcal{D}\psi|^2(m). \end{aligned}$$

The combination of the latter two equations yields that $Hess_m(|\psi|^2)$ is not degenerated if $\mathcal{D}\psi(m)$ is not trivial. Therefore, m is an isolated point of $N(\psi)$. Otherwise, it follows from the last corollary that ψ must be trivial.

□

2.3 The Killing equation

In this section, M^n shall be connected.

Definition 2.10 (Killing spinor fields) *A spinor field $\psi \in \Gamma(S^w)$ is called a Killing spinor field if it satisfies the following differential equation:*

$$\nabla^S \psi = \beta\nu\psi, \quad \beta \in \Gamma(\mathbb{C} \otimes \mathcal{L}^{-1}),$$

where β is the Killing density of ψ.

A non-trivial Killing spinor field vanishes nowhere on a connected manifold, since it is parallel with respect to the covariant derivative $\nabla^S - \beta\nu$. It is obvious, that any Killing spinor field satisfies the twistor equation and can be taken as an eigenspinor of the Dirac operator with the eigen density $-n\beta$. We now investigate the integrability conditions for the existence of non-trivial Killing spinor fields.

Theorem 2.11 *Let $\psi \in \Gamma(S^w)$ be a Killing spinor field.*

1. *β pureley imaginary, $w \neq \frac{n-2}{2}$: $(F \cdot \psi, \psi) = 0$.*
2. *β real: $R = 4n(n-1)\beta^2$.*
   - (a) *$w \neq 0$: W is exact and Einstein-Weyl*
   - (b) *w=0:*
     - i. *$\beta \neq 0$, $n \geq 4$: W is exact and Einstein-Weyl*
     - ii. *$\beta = 0$, $n > 2$, ($4 \neq n$ or M compact): W is closed and Einstein Weyl*
     - iii. *$\beta = 0$, $n = 4$, M non-compact: W is Einstein-Weyl and F is harmonic.*

Remark: For the latter case ($n = 4$ and M^n non-compact) Moroianu gave in [9] an example of a *CSpin* manifold together with a non-closed Weyl structure that carries non-trivial parallel spinor fields.

PROOF:

1. β is purely imaginary: We have
$$R\psi + 2\left(\frac{n-2}{2}+w\right)F\cdot\psi = 4(n-1)n\beta^2\psi - 4(n-1)\nabla\beta\cdot\psi, \tag{13}$$
which itself follows from (11):
$$\frac{n}{4(n-1)}R\psi + \frac{(n-2+2w)n}{4(n-1)}F\cdot\psi = \mathcal{D}^2\psi = -n\mu\nabla^S(\beta\psi) = -n\nabla\beta\cdot\psi + n^2\beta^2\psi.$$
The imaginary part of the product of (13) with ψ is as follows:
$$\left(w+\frac{n-2}{2}\right)(F\cdot\psi,\psi)=0,$$
i.e., we have shown the assertion.

2. β is real: By multiplying (13) with ψ we see that $R = 4n(n-1)\beta^2$ holds.

   $w \neq 0$: We obtain $\nabla_X(\psi,\psi) = (\nabla^S_X\psi,\psi) + (\psi,\nabla^S_X\psi) = \beta(X\cdot\psi,\psi) - \beta(X\cdot\psi,\psi) = 0$, by assumption. Hence, W is exact. This means, that there is a metric g of the conformal class c, for that W is the Levi-Civita connection and W admits a Killingspinor. Therefore, (M^n,g) is Einstein, hence (M^n,c,W) is Einstein-Weyl.

   $w = 0$: By using the definition of Killing spinor fields we obtain: $\mathcal{R}^S\psi = Alt(\nabla\beta)\nu\psi + 2\beta^2(\nu^{21}+c)\psi$, where $X,Y\rfloor\nu^{21}\psi = Y\cdot X\cdot\psi$. This yields:
$$\mu^2\mathcal{R}^S\psi = -n\nabla\beta\otimes\psi - \nabla\beta\cdot\nu\psi - 2\beta^2(n-1)\nu\psi.$$
Together with (7) we obtain:
$$\mu^2 Ric'\psi = 2n\nabla\beta\otimes\psi + 2\nabla\beta\cdot\nu\psi + 4\beta^2(n-1)\nu\psi - \mu^2F\psi - \frac{1}{2}\nu F\cdot\psi. \tag{14}$$
By (13) and the assumption $R = 4n(n-1)\beta^2$ holds and thus we obtain again from (13):
$$F\cdot\psi = -\frac{4(n-1)}{n-2}\nabla\beta\cdot\psi. \tag{15}$$
Inserting (15) into (14) yields:
$$\mu^2 Ric'\psi = 2n\nabla\beta\otimes\psi + 2\nabla\beta\cdot\nu\psi + \frac{R}{n}\nu\psi - \mu^2F\psi + \frac{2(n-1)}{n-2}\nu\nabla\beta\cdot\psi. \tag{16}$$
The operator $\nabla\beta\cdot\nu$ consists of double Clifford products and scalar parts. We now rearrange (16) accordingly.
$$\mu^2 Ric'\psi = 2\left(n-1-\frac{n-1}{n-2}\right)\nabla\beta\otimes\psi + \left(1-\frac{n-1}{n-2}\right)\mu^{12}Alt\nabla\beta\otimes c\psi + \frac{R}{n}\nu\psi - \mu^2F\psi.$$
If we multiply this equation by ψ, we see that $\nabla\beta$ must vanish for $n \neq 3$ since all the other terms are purely imaginary. Therfore, W is exact and as before, W is Einstein-Weyl.

3. The equations (6) and (7) together with the assumption yield
$$\mu^2 Ric'\psi = -\mu^2F\psi - \frac{1}{2}\nu F\cdot\psi.$$
Then (13), $R = 0$ and the assumption impose $F\cdot\psi$ to vanish. Since ψ vanishes nowhere however, $Ric' = -F$. Therefore, the symmetric part of Ric' reduces to its trace (which is 0), i.e., W is Einstein-Weyl. Hence, Theorem 3.6 in [2] is applicable, which yields all remaining assertions.

□

2.4 Two dimensional examples

1. *Killing spinor fields of weight* $\frac{1}{2}$:
   We can find imaginary Killing spinor fields ψ of weight $\frac{1}{2}$ on $(\mathbb{R}^2, [g], x_1 dx^2)$. Because of Theorem 1.2 they have to be a solution of

$$X(\psi) = \frac{1}{2} X \cdot \theta \cdot \psi + \beta_g X \cdot \psi = X \cdot \left(\frac{1}{2} x_1 \partial_2 + \beta_g \right) \cdot \psi,$$

   where $X \in \mathbb{R}^2$ and $\Gamma(\mathbb{C} \otimes \mathcal{L}^1) \ni \beta = \beta_g l_g$ hold true. (∂_1, ∂_2) are said to be the standard basis in $\mathbb{R}^2$. If one uses the following representation of the Clifford algebra

$$\partial_1 \longmapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \partial_2 \longmapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

   one obtains

$$X \cdot \begin{pmatrix} \beta_g & \frac{i}{2} x_1 \\ \frac{i}{2} x_1 & \beta_g \end{pmatrix} \psi_0 = 0.$$

   We find a non-trivial kernel of the matrix for all X if and only if $\beta_g = \pm \frac{i}{2} x_1$. An element of this kernel must be of the form $\psi_0 = \begin{pmatrix} a \\ \mp a \end{pmatrix}$ with $a \in \mathbb{C}$. We obtain, just as stated in Theorem 2.11:

$$(F\psi_0, \psi_0) = x_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ \mp a \end{pmatrix} \cdot \begin{pmatrix} a \\ \mp a \end{pmatrix} = x_1 \begin{pmatrix} \pm a \\ a \end{pmatrix} \cdot \begin{pmatrix} a \\ \mp a \end{pmatrix} = 0.$$

2. *Parallel spinor fields of weight* 0*:*
   On $(\mathbb{R}^2, [g], x_1 dx^2)$ we have to solve:

$$X(\psi) = \frac{1}{2} X \cdot \theta \cdot \psi + \frac{1}{2} \theta(X) \psi = \frac{1}{2} X_1 x_1 \partial_1 \cdot \partial_2 \cdot \psi$$

   for $\psi \in \Gamma(S^0)$, where $X = \sum_{i=1}^2 X_i \partial_i$.
   We use the following representation of the Clifford algebra

$$\partial_1 \longmapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \partial_2 \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

   Therfore

$$\partial_1 \cdot \partial_2 \longmapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

   and so we are given

$$X(\psi^+) = \frac{i}{2} X_1 x_1 \psi^+, \quad X(\psi^-) = -\frac{i}{2} X_1 x_1 \psi^-,$$

   where ψ^+ and ψ^- correspond to the splitting $\Delta_2 = \Delta_2^+ \oplus \Delta_2^-$. It is, however, not difficult to determine the solution of this system.

$$\psi^+(x) := \exp\left(\frac{i}{4} x_1^2 \right) \psi_0^+, \quad \psi^-(x) := \exp\left(-\frac{i}{4} x_1^2 \right) \psi_0^-,$$

   where $\psi_0^\pm \in \mathbb{C}$.

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